Abstract

In this research I studied the Heawood Conjecture (map color theorem), given by Percy John Heawood in 1890 and, for most cases, proved by Gerhard Ringel and Ted Youngs in 1968. Heawood conjectured that the minimum number of colors necessary to color all graphs drawn on a closed surface S is equal to

$$\chi := \left\lfloor \frac{7 + \sqrt{49 - 24E(S)}}{2} \right\rfloor$$

where E is the Euler characteristic of the surface.

I first study the proof of the Heawood inequality, which proves that the chromatic number is less than or equal to χ. It can be proved that $E(S) \leq 2$, and by considering $E \leq 0$, $E = 1$, $E = 2$ (the four color theorem, proved with help of computer later), the inequality is proved.

Abstract, conti.

Then, I study the proof that this upper bound is in fact a least upper bound for most cases. This is proved by considering separate cases (orientable: $1, 4, 9; 2, 8, 11$; non-orientable: index $1, 2, 3$) for orientable and non-orientable surfaces, along with induction.

At the end of the research, I gave presentation on what I learned, especially on the only exception of the conjecture. It is the Klein bottle, i.e. the non-orientable surface of genus 2. The floor function of the Heawood inequality gives 7 while it can be proved that $\chi(S) = 6$. First there are examples (K_6 embedding into N_2 possible) coloring a graph on N_2 which needs at least 6 colors, so $\chi(S) \geq 6$, but it can be proved that K_7 cannot be embedded into the surface of N_2. Then it suffices to show that graphs can be embedded in N_2 with chromatic number 7 do not exist. It can then be proved by contradiction that if so, then the graph is hexavalent, which can then be colored with 6 colors.

Klein Bottle: an exception of the Heawood Conjecture

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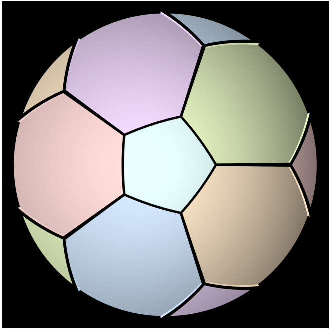
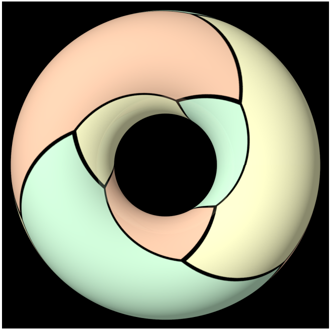
Preliminaries: Graph Embedding





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Heawood Inequality

Let S be a closed surface. Then

$$\chi(S) \leq \left\lfloor \frac{7+\sqrt{49-24E(S)}}{2} \right\rfloor$$

where

$\chi(S) := \max_{G \subset S} \chi(G)$ is the chromatic number, i.e. the minimum number of colors necessary to color all graphs drawn on S;

$E(S) := \alpha_0 - \alpha_1 + \alpha_2$ is the Euler characteristic of the surface. $\alpha_0, \alpha_1, \alpha_2$ are numbers of vertices, edges, and polygons.

A Sketch of Proof

WLOG, assume that G is critical (otherwise, pick one of its critical subgraph).

Lemma

$(\chi(G) - 1)\alpha_0 \leq 2\alpha_1$ for G critical.

Lemma

$\alpha_1 \leq 3\alpha_0 - 3E(S)$ for G embedded into a closed surface S.

Then, one has

$$\chi(G) - 1 \leq 6 - \frac{6E(S)}{\alpha_0}$$

A Sketch of Proof

From the fact that each polyhedron P has $E(P) \leq 2$, it suffices to consider three cases:

$E(S) \leq 0$: solving $\chi^2 - 7\chi + 6E \leq 0$;

$E(S) = 1$: noting that χ is an integer;

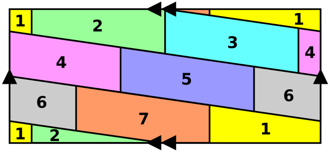
$E(S) = 2$, which implies that S is a sphere, follows from the *four color theorem* which is proved much later with the help of computer exhausting all possible cases.

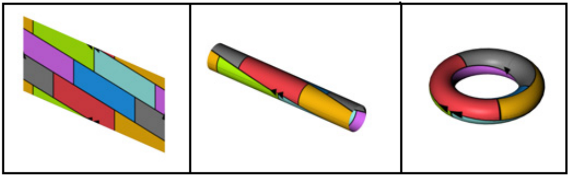
So we're done.

The Conjecture

$$\chi(S) = \left\lfloor \frac{7 + \sqrt{49 - 24E(S)}}{2} \right\rfloor ?$$

For example, a torus (T) has $E(T) = 0$, and $\chi(T) = 7$. Here is a construction:





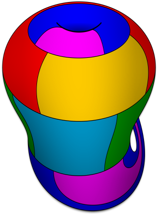
The Conjecture

The conjecture fails with only one exception: the *Klein Bottle* (N_2).

Note that $E(N_2) = 0$, but

$$\chi(N_2) = 6 \neq \left\lfloor \frac{7 + \sqrt{49 - 24E(N_2)}}{2} \right\rfloor = 7$$

For example, here is a picture of using 6 colors to color a graph on N_2 (though we need to show that $\nexists G$ that needs at least 7 colors).



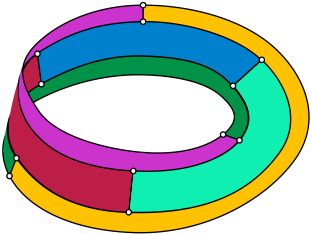
(source)

A Sketch of Proof

Lemma

The complete graph K_7 cannot be embedded into the surface of N_2.

A simple fact: $\chi(N_2) \geq 6$, since K_6 can be embedded into N_1, and so N_2.



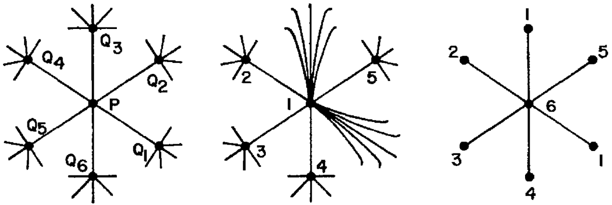
Example: the Franklin graph embedded in the Klein bottle. (source)

A Sketch of Proof

So it suffices to show that there does not exist a graph embedded in N_2 with chromatic number 7.

Suppose there exists such G: then G is *hexavalent*.

But then, we can color G with 6 colors.



So we're done.