Preparation for Quantitative Finance Interviews

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Theorem 2.9 (Stokes). *Let Σ be a smooth oriented surface in $\mathbb{R}^3$ with boundary $\partial\Sigma \equiv \Gamma$. If a vector field $\mathbf{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$ is defined and has continuous first order partial derivatives in a region containing Σ, then*

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{\Gamma}$$

More explicitly, the equality says that

$$\iint_\Sigma \left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathrm{d}z\, \mathrm{d}x + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y \right)$$
$$= \oint_{\partial\Sigma} (F_x\, \mathrm{d}x + F_y\, \mathrm{d}y + F_z\, \mathrm{d}z)\,.$$

Note that *Green's theorem* is the two-dimensional special case of Stokes' theorem.

3 Functions of a complex variable

Reference text: [Ahl79].

The formal definition of a complex number $z = x + yi = (x, y) \in \mathbb{C} := \mathbb{R}^2$ is to consider the field $(\mathbb{R}^2, +, \cdot)$ with addition and multiplication

$$(x, y) + (u, v) = (x + u, y + v), \qquad (x, y) \cdot (u, v) = (xu - yv, xv + yu)$$

with neutral elements $0_\mathbb{C} = (0, 0)$ for $+$ and $1_\mathbb{C} = (1, 0)$ for $\cdot$.

Note that $\mathbb{C}$ cannot be ordered. However, with modulus defined as $|z| := \sqrt{x^2 + y^2} \in [0, \infty)$ and conjugate defined as $\overline{z} := x - yi$, we have the analogous triangle inequalities

$$|z + w| \le |z| + |w|, \qquad |z - w| \ge ||z| - |w||$$

and Cauchy-Schwarz inequality

$$\left| \sum_{j=1}^n \overline{z_j} w_j \right| = \left| \sum_{j=1}^n z_j w_j \right| \le \sqrt{\sum_{j=1}^n |z_j|^2} \sqrt{\sum_{j=1}^n |w_j|^2}$$

mentioned earlier in §1.

A complex number $z = a + bi$ can be expressed by *polar coordinates*:

$$a + bi = r \cos\theta + i \cdot r \sin\theta$$

where $\arg(z) := \theta \in (-\pi, \pi]$. Upon multiplication of two complex numbers, the moduli are multiplied, and the arguments are added.

Exercise 3.1 ([SRW19], 1.4). *Solve $x^6 = 64$.*

Solution. The moduli should always be 2, and the argument θ can be any value such that

$$6\theta = 2k\pi, \qquad k \in \mathbb{N}_0, \theta \in [0, 2\pi)$$

Therefore, there are six possible solutions:

$$\boxed{2\cos(k\pi/3) + 2i\sin(k\pi/3), \qquad k \in [\![0, 5]\!]}$$

The extension of functions $e^z, \log z, \sin z, \cos z$, etc. should be natural in the sense that many of the familiar properties of $\sin, \cos, \exp, \log$ are retained. We define the complex *exponential function* as

$$e^{a+bi} := e^a \cdot (\cos b + i \sin b), \qquad z = a + bi \in \mathbb{C}$$

and the complex sine and cosine functions as

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad z \in \mathbb{C}$$

Exercise 3.2 ([SRW19], 1.1). *Calculate i^i.*

Solution. Note that $i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\cdot\pi/2}$. Therefore

$$i^i = \left(e^{i\cdot\pi/2}\right)^i = \boxed{e^{-\pi/2}}$$

A solution z of the equation $e^z = w$ is called a *logarithm* of w, denoted $z = \log w$. Every $w \in \mathbb{C}\backslash\{0\}$ has countably many logarithms:

$$\log(w) = \log|w| + i \cdot (\arg(w) + 2\pi n), \qquad n \in \mathbb{Z},$$

and the principal value of the logarithm of w is set for $n = 0$.

4 Ordinary differential equations

Solvable first-order ODEs

1. Separable:

$$x' = f(x)g(t) \qquad \Longrightarrow \qquad \int \frac{1}{f(x)}\,\mathrm{d}x = \int g(t)\,\mathrm{d}t + C$$

Exercise 4.1 ([SRW19], 1.14). *Find $f(x)$ such that*

$$f'(x) = f(x)(1 - f(x))$$

Solution. Let $y = f(x)$. Then the equation is separable.

2. Homogeneous of degree k for some arbitrary value a:

$$f(at, ax) = a^k f(t, x)$$

For example, take $k = 1, a = 1/t$:

$$x' = f(x/t)$$

Let $y = x/t$, we get

$$y' \cdot t + y = f(y)$$

which reduces to case 1, a separable equation.

3. More generally, consider

$$x' = f\left(\frac{ax + bt + c}{\alpha x + \beta t + \gamma}\right)$$

where $a, b, c, \alpha, \beta, \gamma$ are constants.

a) If $c = \gamma = 0$, rewrite it as

$$x' = f\left(\frac{ax/t + b}{\alpha x/t + \beta}\right)$$

which reduces to case 2.

b) If $c, \gamma \neq 0$, but $a/\alpha = b/\beta = 1/k$, we let $y = ax + bt$, then (noting that a is constant so $a' = 0$)

$$y' = a'x + ax' + b = af\left(\frac{y + c}{ky + \gamma}\right) + b$$

which reduces to case 1, a separable equation.

c) If $c, \gamma \neq 0$ and $a/\alpha \neq b/\beta$, we solve the system

$$\begin{cases} ax + bt + c & = 0 \\ \alpha x + \beta t + \gamma & = 0 \end{cases}$$

which must have a solution, say x_0, t_0. Take $y = x - x_0, s = t - t_0$, and we have

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}t} = f\left(\frac{ax + bt + c - 0}{\alpha x + \beta t + \gamma - 0}\right) = f\left(\frac{ay + bs}{\alpha y + \beta s}\right)$$

which reduces to case 3(a).

4. Exact:
$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$
is exact if there exists a function $g(x,y)$ such that
$$\mathrm{d}g = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y, \qquad M(x,y) = \frac{\partial g}{\partial x}, N(x,y) = \frac{\partial g}{\partial y}$$
This happens if and only if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Then $g(x,y) = C$ where C is some constant is the solution to this equation.

5. Integrating factors: if $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$ is not exact but $I(x,y)\,(M\,\mathrm{d}x + N\,\mathrm{d}y)$ is, I is called an *integrating factor*.

   a) If
   $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =: g(x)$$
   is a function of x alone, then
   $$I(x,y) = \exp\left(\int g(x)\,\mathrm{d}x\right)$$

   b) If
   $$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =: h(y)$$
   is a function of y alone, then
   $$I(x,y) = \exp\left(-\int h(y)\,\mathrm{d}y\right)$$

   Then it is reduced to case 4.

6. Linear nonhomogeneous equation:
$$x'(t) = k(t)x + a(t)$$
We multiply each side by
$$K(t) := \exp\left\{-\int_{t_0}^{t} k(s)\,\mathrm{d}s\right\}$$
Then
$$K(t)\left(x' - kx\right) = (Kx)' \qquad \Longrightarrow \qquad \int \mathrm{d}(Kx) = \int K(t)a(t)\,\mathrm{d}t + C$$

which yields the result

$$x(t) = \frac{1}{K(t)} \left(\int_{t_0}^{t} K(s)a(s)\,\mathrm{d}s + x(t_0) \right)$$

7. Some nonlinear first-order ODEs:

   a) Bernoulli:

   $$x'(t) = F_t x + g(t)x^n, \qquad n \neq 0, 1$$

   We let $y = x^{1-n}$ and have

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = (1-n)x^{-n}x' = (1-n)x^{-n}\left(f(t)x + g(t)x^n\right) = (1-n)\left(fy + g\right)$$

   b) Ricatti:

   $$x'(t) = f(t)x + g(t)x^2 + h(t)$$

   Suppose we already has a particular solution $p(t)$ so that $p' = fp + gp^2 + h$. Subtract it from the original equation and let $y = x - p$:

   $$\begin{aligned} x' - p' &= f(x-p) + g(x+p)(x-p) \\ y' &= fy + g(y+2p)y = \left(f(t) + g(t)2p(t)\right)y(t) + g(t)y^2(t) \end{aligned}$$

   which reduces to a Bernoulli equation.

Second-order linear ODEs

We start by considering a_0, a_1 real constants for

$$y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

Solve the the *characteristic function*

$$\lambda^2 + a_1\lambda + a_0 = 0$$

and get two solutions λ_1, λ_2.

- Case 1: if $\lambda_1, \lambda \in \mathbb{R}, \lambda_1 \neq \lambda_2$:

  $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

- Case 2: if $\lambda_{1,2} = a \pm bi, b \neq 0$ (noting that roots must be conjugate pairs):

  $$y = d_1 e^{(a+bi)x} + d_2 e^{(a-bi)x} = e^{ax} \cdot \left(c_1 \cos(bx) + c_2 \sin(bx)\right)$$

- Case 3: if $\lambda_1 = \lambda_2 = \lambda$:

  $$y = (c_1 + c_2 x)e^{\lambda x}$$

Exercise 4.2 (Baruch). *Solve* $4y'' - 4y' + y = 0$ *with initial conditions* $y(0) = 1, y'(0) = 0$.

Solution. Consider $4\lambda^2 - 4\lambda + 1 = 0$ whose solutions are $\lambda_{1,2} = 1/2$. It is a Case 3 scenario, and thus we let

$$y = (c_1 + c_2 x)e^{x/2}$$

Plugging the initial conditions, we get $c_1 = 1, c_2 = -1/2$.

Now, suppose $a_1 = p(x), a_0 = q(x)$, i.e. we consider

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

- Case 1: Suppose p, q are analytic, then we can let $y(x) = \sum_{n=0}^{\infty} A_k x^k$ and apply the *power series method.*

- Case 2: A *Cauchy-Euler* equation has the form

  $$t^2 y'' + aty' + by = 0 \qquad \iff \qquad y'' + \frac{ay'}{t} + \frac{by}{t^2} = 0$$

  We assume $y = t^m$ and plugging in we get $m^2 + (a-1)m + b = 0$. Suppose this equation has two solutions $m_{1,2}$.

  - if $m_1 \neq m_2 \in \mathbb{R}$, $y = c_1 t^{m_1} + c_2 t^{m_2}$.
  - if $m_1 \neq m_2 \in \mathbb{C}$, they are conjugates, so we let $m_{1,2} = \alpha \pm \beta i$ and have

    $$y = c_1 \cdot t^\alpha \cos(\beta \log t) + c_2 \cdot t^\alpha \sin(\beta \log t)$$

  - if $m_1 = m_2 = m$, they can only be reals, and we have

    $$y = c_1 \cdot t^m \log t + c_2 \cdot t^m$$

    Indeed, if it is an ODE with higher order, and m becomes a root with multiplicity k, then the basis solutions are:

    $$t^m, t^m \log t, t^m (\log t)^2, t^m (\log t)^3, \ldots, t^m (\log t)^{k-1}$$

- Case 3: the *method of Frobenius*: the differential equation

  $$w'' + p(z)w' + q(z)w = 0$$

  has an *isolated singular point* at the origin if the coefficients p and q are analytic and single-valued in a disk $|z| < R$ except at $z = 0$ (analytic in the punctured disk). The origin is a *regular singular point* if p has a pole of order at most one and q a pole of order at most two there. In other words, if the origin is a regular singular point then $p(z) = z^{-1}P(z)$ and

$q(z) = z^{-2}Q(z)$ where P and Q are analytic and single-valued in the full disk, including the origin. We'll write the standard equation with a regular singular point at the origin in the form

$$z^2 w'' + zP(z)w' + Q(z)w = 0.$$

The functions P and Q then have power-series expansions

$$P(z) = \sum_{k=0}^{\infty} P_k z^k, \quad Q(z) = \sum_{k=0}^{\infty} Q_k z^k$$

convergent in this disk. Then we seek a solution of the form

$$w(z) = z^{\mu} \sum_{k=0}^{\infty} a_k z^k$$

Now suppose the equation is nonhomogeneous:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \qquad r(x) \neq 0$$

First we solve the homogeneous equation by the methods above

$$y''(x) + p(x)y'(x) + q(x)y(x)$$

and get a basis of two solutions $b_1(x), b_2(x)$. We calculate the Wronskian:

$$W(x) := \begin{vmatrix} b_1 & b_2 \\ b_1' & b_2' \end{vmatrix}$$

and a particular solution $p(x)$ is given as

$$p(x) = b_1 v_1 + b_2 v_2 \qquad \text{where} \qquad v_1 = -\int \frac{r(x)b_2(x)}{W(x)}\,\mathrm{d}x, v_2 = \int \frac{r(x)b_1(x)}{W(x)}\,\mathrm{d}x$$

Therefore, a general solution is given by

$$y(x) = C_1 b_1(x) + C_2 b_2(x) + p(x)$$

Exercise 4.3 ([SRW19], 1.13)**.** *Solve* $y'' - 4y' + 4y = 1$.

Solution. Solving $\lambda^2 - 4\lambda + 4 = 0$ gives us $\lambda_{1,2} = 2$. Now, since RHS $= 1$, a particular solution is easy to guess: $y_p = 1/4$. Therefore, a general solution is given by

$$y(x) = c_1 e^{2x} + c_2 e^{2x} x + 1/4$$

for some constants c_1, c_2.

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